On (a, b)-step Hamiltonian graphs

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Abstract

For integers $a, b \geq 1$, a (p, q)-graph $G = (V, E)$ is said to have an $AL(a, b)$-step traversal if its vertices $v_1, v_2, \ldots, v_p$ are such that for each odd $i = 1, 3, 5, \ldots$, the distance between v_i and v_{i+1} is equal to a and for each even $i = 2, 4, 6, \ldots$, the distance between v_i and v_{i+1} is equal to b. A graph that admits an $AL(a, b)$-step traversal is called an (a, b)-traceable graph. The sequence $v_1, v_2, \ldots, v_p$ is called an (a, b)-step Hamiltonian path. A Hamiltonian graph can be considered as a (1,1)-step Hamiltonian graph. In general, if $d(v_1, v_p) = b$ (for even p) or $d(v_1, v_p) = a$ (for odd p), we say G is (a, b)-step Hamiltonian. In this paper, we give some ways to construct new (a, b)-step Hamiltonian graphs. We also consider (a, b)-Hamiltonicity of paths, cycles and some related graphs. As a consequence, we show that for any $b > a \geq 1$ there exist an infinite number of (a, b)-step Hamiltonian graphs.

Keywords: Hamiltonian, $AL(a, b)$-step traversal, (a, b)-step Hamiltonian.

AMS 2000 MSC: 05C78, 05C25

1 Introduction

The problem of hamiltonicity of graphs has been studied extensively since Tait's attempt to prove every 3-connected planar cubic graph has a Hamiltonian cycle (see [4]). In [1], the authors introduced the concept of k-step Hamiltonian graphs (also see [2, 3]).

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Definition 1.1 *For an integer $k \geq 1$, a (p,q)- graph $G = (V,E)$ is said to admit an $AL(k)$-step traversal if its vertices $v_1, v_2, \ldots, v_p$ are such that for each $i = 1, 2, \ldots, p-1$, the distance between v_i and v_{i+1} is equal to k. The sequence $v_1, v_2, \ldots, v_p$ is called a k-step Hamiltonian path and G is said to be k-traceable. A graph G is k-step Hamiltonian if it is k-traceable with $d(v_p, v_1) = k$.*

Clearly, a 1-step Hamiltonian graph is also a Hamiltonian graph. In [1], it was shown that all bipartite graphs are not 2-step Hamiltonian, and many families of graphs that are or are not k-step Hamiltonian for certain or all $k \geq 2$ are also given. In this paper, we extend the concept of k-step Hamiltonian to (a,b)-step Hamiltonian.

Definition 1.2 *For integers $a, b \geq 1$, a (p,q)- graph $G = (V,E)$ is said to admit an $AL(a,b)$-step traversal if its vertices $v_1, v_2, \ldots, v_p$ are such that for each odd $i = 1, 3, 5, \ldots$, the distance between v_i and v_{i+1} is equal to a and for each even $i = 2, 4, 6, \ldots$, the distance between v_i and v_{i+1} is equal to b. A graph that admits an $AL(a,b)$-step traversal is called an (a,b)-traceable graph. The sequence $v_1, v_2, \ldots, v_p$ is called an (a,b)-step Hamiltonian path. In general, if $d(v_1, v_p) = b$ (for even p) or $d(v_1, v_p) = a$ (for odd p), we say G is (a,b)-step Hamiltonian.*

Example 1.1 *The graph in Figure 1 is not Hamiltonian, k-step Hamiltonian nor (1,2)-step Hamiltonian but is (1,2)-traceable.*

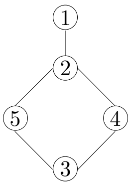


Figure 1: An AL(1,2)-step traversal graph

Example 1.2 *The graph in Figure 2 is not Hamiltonian but is 2-step Hamiltonian and (1,2)-step Hamiltonian.*

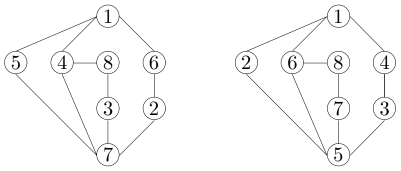


Figure 2: Graph with a (a) 2-step Hamiltonian, (b) (1,2)-step Hamiltonian tour

Clearly, (a, a)-step Hamiltonian is a-step Hamiltonian and a 1-step Hamiltonian graph is also Hamiltonian. Moreover, if a graph of even order is (a, b)-step Hamiltonian, then it is necessarily (b, a)-step Hamiltonian. In such a case, we may assume that $b > a \geq 1$. In this paper, we show that an (a, b)-step Hamiltonian graph exists for all a, b and give some ways to construct new (a, b)-step Hamiltonian graphs. We also consider (a, b)-Hamiltonicity of cycles and some related graphs. As a consequence, we showed that there exist infinitely many 1-connected $(a, a+1)$-step Hamiltonian graphs.

Proposition 1.1 *If G is (a, b)-step Hamiltonian of even order, then for every vertex v of G, there exist 2 distinct vertices x and y such that $d(v, x) = a$ and $d(v, y) = b$.*

Definition 1.3 *For $k > 2$, let $D_k(G)$ be the distance k graph generated by G with vertices in $D_k(G)$ adjacent if and only if the distance between them in G is k.*

The following theorem follows directly.

Theorem 1.2 *If a graph G of even order p is (a, b)-step Hamiltonian, then $D_a(G)$ and $D_b(G)$ each contains a perfect matching subgraph, say $M_a(G)$ and $M_b(G)$ respectively, such that $M_a(G) \cup M_b(G)$ contains an induced cycle C_p.*

Corollary 1.3 *If G is a graph of even order p such that $D_a(G)$ or $D_b(G)$ (a) has an odd order component; or (b) has less than $p/2$ edges, then G is not (a, b)-step Hamiltonian.*

2 Existence and Construction of (a, b)-step Hamiltonian graphs

Theorem 2.1 *There exist $(k, k+1)$-step Hamiltonian graphs for each integer $k \geq 1$.*

Proof. A $(k, k+1)$-step Hamiltonian graph is constructed by joining two paths of order $2k+2$ given by $u_1, u_2, u_3, \ldots, u_{2k+2}$ and $v_1, v_2, v_3, \ldots, v_{2k+2}$ and joining vertices u_1 and v_{k+3} by an edge and v_1 and u_{k+3} by an edge. A $(k, k+1)$-step cycle is $u_{k+2}, u_2, u_{k+3}, u_3, \ldots, u_{2k+2}, v_1, v_{k+2}, v_2, v_{k+3}, \ldots, v_{2k+2}, u_1, u_{k+2}$.

Example 2.1 *In Figure 3, we give the graph described in Theorem 2.1 and the constructed cycle for $k = 2, 3$. Note that for $k = 1$, we get a (1,2)-step Hamiltonian cycle C_8.*

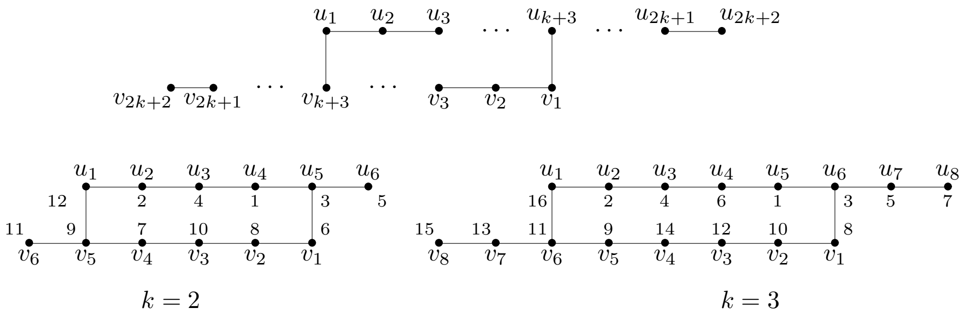


Figure 3: Construction of $(k, k+1)$-step Hamiltonian graphs

More generally, we have

Theorem 2.2 *For any $b > a \geq 1$, an infinite number of (a, b)-step Hamiltonian graphs exist.*

Proof. Choose a cycle C_n such that $gcd(n, a+b-2) = 1$ and $n \geq 2(b-1)$. Let $C_n = u_1 u_2 u_3 \cdots u_n u_1$. Attach a degree 1 vertex v_i by an edge to the vertex u_i of C_n. An (a, b)-step Hamiltonian tour is then given by $u_1, v_a, u_{a+b-1}, v_{2a+b-2}, u_{2a+2b-3}, \cdots, u_{n-a-b+3}, v_{n-b+2}, u_1$ with the subscripts taken over mod n. Since there are an infinite number of n such that $gcd(n, a+b-2) = 1$ and $n \geq 2(b-1)$, there are in infinite number of cycles C_n which are (a, b)-step Hamiltonian. ☐

Example 2.2 *Let $a = 3, b = 8$, then $n \geq 2(8-1) = 14$. Take $n = 16$, for example, such that $gcd(16, 3+8-2) = 1$. A $(3, 8)$-step Hamiltonian tour is given by u_1, v_3, u_{10}, v_{12}, u_3, v_5, u_{12}, v_{14}, u_5, v_7, u_{14}, v_{16}, u_7, v_9, u_{16}, v_2, u_9, v_{11}, u_2, v_4, u_{11}, v_{13}, u_4, v_6, u_{13}, v_{15}, u_6, v_8, u_{15}, v_1, u_8, v_{10}, u_1.*

Corollary 2.3 *For any $b > a \geq 1$, an (a, b)-step Hamiltonian $C_n \times P_2$ exists.*

We shall now present a few methods for constructing other infinitely many (a, b)-step Hamiltonian graphs. In what follows, we only consider even order graphs, unless specified otherwise.

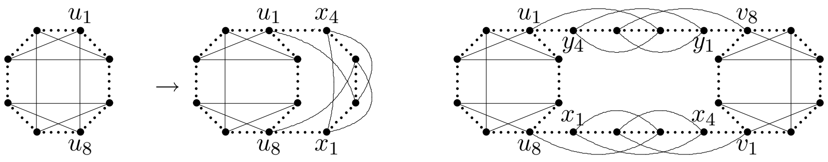
For $a \geq 1$, let G_1 and G_2 be two $(a, a+1)$-step Hamiltonian graphs of order s and t with an $(a, a+1)$-step Hamiltonian tour $u_1, u_2, u_3, \ldots, u_{s-1}, u_s, u_1$, and $v_1, v_2, v_3, \ldots, v_{t-1}, v_t, v_1$, respectively. Let $x_1, x_2, \ldots, x_{2a}$ and $y_1, y_2, \ldots, y_{2a}$ be 2 distinct paths of length $2a$.

Denote by $G_1(a, a+1)$ (respectively, $G_2(a, a+1)$) the graph obtained from G_1 (respectively, G_1 and G_2) by joining x_1 to u_s and x_{2a} to u_1 (respectively, x_1 to u_s, x_{2a} to v_1, y_1 to v_t, and y_{2a} to u_1).

Theorem 2.4 *The graphs $G_1(a, a+1)$ and $G_2(a, a+1)$ are $(a, a+1)$-step Hamiltonian.*

Proof. Observe that the graph $G_1(a, a+1)$ has an $(a, a+1)$-step Hamiltonian tour given by $u_1, u_2, u_3, \ldots, u_{s-1}, u_s, x_{a+1}, x_1, x_{a+2}, x_2, \ldots, x_{2a-1}, x_{a-1}, x_{2a}, x_a, u_1$. Similarly, an $(a, a+1)$-step Hamiltonian tour for $G_2(a, a+1)$ is given by $u_1, u_2, u_3, \ldots, u_{s-1}, u_s, x_{a+1}, x_1, x_{a+2}, x_2, \ldots, x_{2a-1}, x_{a-1}, x_{2a}, x_a, v_1, v_2, v_3, \ldots, v_{t-1}, v_t, , y_{a+1}, y_1, y_{a+2}, y_2, \ldots, y_{2a-1}, y_{a-1}, y_{2a}, y_a, u_1$. $\square$

Example 2.3 *In Figure 4, we give a $(2, 3)$-step Hamiltonian tour for the graphs $G_1(2, 3)$ and $G_2(2, 3)$ constructed from a $(2, 3)$-step Hamiltonian C_8.*



$G_1(2, 3)$ $G_2(2, 3)$

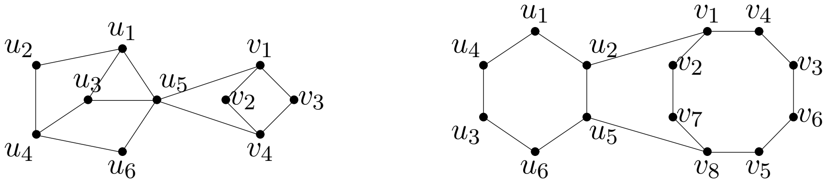
Figure 4: New (2,3)-step Hamiltonian graph $G_1(2, 3)$ and $G_2(2, 3)$ constructed from C_8

For $b > a \geq 1$, let G_1 and G_2 be two (a, b)-step Hamiltonian graphs of order s and t respectively with an (a, b)-step Hamiltonian tour given by $u_1, u_2, \ldots, u_s, u_1$ and $v_1, v_2, \ldots, v_t, v_1$, respectively. Let $u_1, x, \ldots, y, u_s$ be a shortest $u_1 - u_s$ path (of length b) in G_1. Construct a new graph $G(a, b)$ from G_1 and G_2 by adding an edge joining vertex x to v_1, and an edge joining vertex y to v_t. Note that $x = y$ if $a = 1, b = 2$, and hence $G(1, 2)$ is 1-connected.

Theorem 2.5 *For $b > a \geq 1$, the graph $G(a, b)$ is (a, b)-step Hamiltonian.*

Proof. Observe that $d(u_1, v_t) = d(u_s, v_1) = b$. An (a, b)-step Hamiltonian tour in $G(a, b)$ is given by $u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_t, u_1$. $\square$

Example 2.4 *In Figure 5, we give a $(1, b)$-step Hamiltonian tour in $G(1, b)$ constructed from 2 distinct $(1, b)$-step Hamiltonian graphs for $b = 2, 3$.*



$G(1, 2)$ $G(1, 3)$

Figure 5: New $(1, b)$-step Hamiltonian graphs for $b = 2, 3$

It is well known that all Hamiltonian graphs are 2-connected. However, the next theorem shows that there are infinitely many 1-connected $(a, a+1)$-step Hamiltonian graphs for $a = 1, 2$.

Theorem 2.6 *There exist 1-connected* $(a, a+1)$*-step Hamiltonian graphs of even order at least* $4a+2, a=1,2$.

Proof. A construction method for $a=1,2$ is given in Figures 6 and 7, respectively. ☐

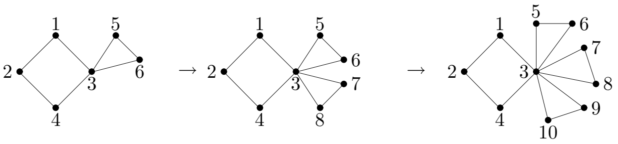


Figure 6: 1-connected (1,2)-step Hamiltonian graphs of even order ≥ 6

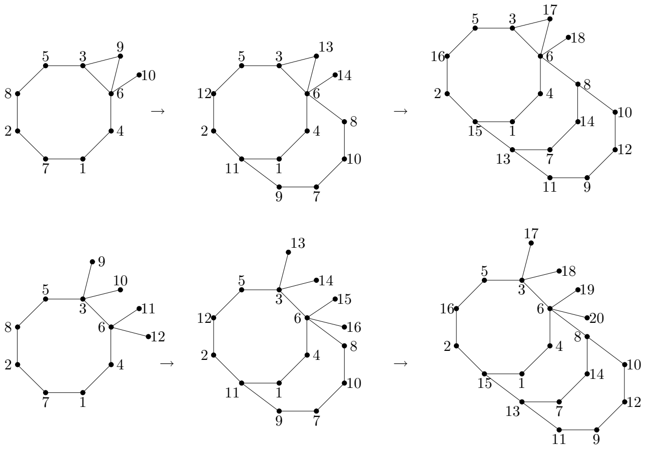


Figure 7: 1-connected (2,3)-step Hamiltonian graphs of even order ≥ 10

3 Paths

The simplest connected graph is the path P_n with n vertices. Let the vertices be labeled $0, 1, 2, \ldots, n-1$ consecutively.

Theorem 3.1 *All paths* P_n *are* $(1,2)$*-step traceable.*

Proof. Label the vertices of P_n as $0, 1, 2, ..., n-1$. For $n = 4k, 4k+1$, or $4k+2$ we use the sequence $0, 1, 3, 2, 4, 5, 7, 6, 8, ..., 4k-4, 4k-3, 4k-1, 4k-2, 4k$, with last vertex in the sequence $4k-1, 4k$ or $4k+1$ respectively. See Figure 8. ☐

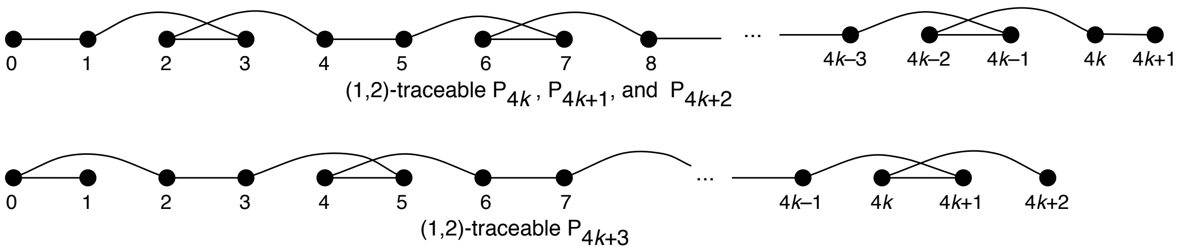


Figure 8: (1,2) traceable paths

Theorem 3.2 *Necessary conditions for P_n to be (a, b)-step Hamiltonian are $n = 2ab$ and $gcd(a, b) = 1$. Any (a, b)-step Hamiltonian cycle will be composed of the edges in the set*

$E = \{(0, a), (1, a + 1), (2, a + 2), \ldots, (a - 1, 2a - 1)\}$

$\cup \{(2a, 3a), (2a + 1, 3a + 1), (2a + 2, 3a + 2), \ldots, (3a - 1, 4a - 1)\}$

$\ldots$

$\cup \{(n - 2a, n - a), (n - 2a + 1, n - a + 1), (n - 2a + 2, n - a + 2), \ldots, (n - a - 1, n - 1)\}$

$\cup \{(0, b), (1, b + 1), (2, b + 1), \ldots, (b - 1, 2b - 1)\}$

$\cup \{(2b, 3b), (2b + 1, 3b + 1), (2b + 2, 3b + 2), \ldots, (3b - 1, 4b - 1)\}$

$\ldots$

$\cup \{(n - 2b, n - b), (n - 2b + 1, n - b + 1), (n - 2b + 2, n - b + 2), \ldots, (n - b - 1, n - 1)\}$.

Proof. $gcd(a, b) = 1$. Let the vertices be labeled $0, 1, 2, \ldots, n - 1$. Two vertices u and v are connected in $D_a(P_n) \cup D_b(P_n)$ only if $v = u + ka + mb$ for some integers k and m. Because $ka + mb$ is a multiple of $d = gcd(a, b)$, we must have $u \equiv v \pmod{gcd(a, b)}$. Therefore P_n can only be (a, b)-step Hamiltonian if $d = 1$.

$n = 2ab$. Each vertex in an (a, b)-step Hamiltonian cycle must have one edge at distance a and one at distance b within that cycle. Vertices $0, 1, 2, \ldots, a - 1$ can only have their length a edges connected to vertices $a, a+1, a+2, \ldots, 2a-1$ respectively. Similarly, vertices $2a, 2a+1, \ldots, 3a-1$ only have length a edges to vertices $3a, 3a + 1, \ldots, 4a - 1$. By induction, we see that n must be a multiple of $2a$. Similarly, n must be a multiple of $2b$, and since $gcd(a, b) = 1$, the smallest such n is $2ab$. If $n = 2kab$ for $k > 1$, then $D_a(P_n) \cup D_b(P_n)$ is composed of k components each of order $2ab$, so P_n cannot be (a, b)-step Hamiltonian. □

However, $gcd(a, b) = 1$ and $n = 2ab$ is not sufficient for P_n to be (a, b)-step Hamiltonian.

Example 3.1 *$P_{2(2k+1)}$ is $(1, 2k + 1)$-step Hamiltonian with vertex sequence:* $0, 1, a + 1, a + 2, 2, 3, a+3, a+4, 4, 5, a+5, a+6, \ldots, a-3, a-2, 2a-2, 2a-1, a-1, a, 0$. *However, $D_1(P_{2(2k)}) \cup D_{2k}(P_{2(2k)}) \cong k \times C_4$.*

For example, P_{10} is $(1, 5)$-step Hamiltonian with vertex sequence $0, 1, 6, 7, 2, 3, 8, 9, 4, 5, 0$.

P_{12} is not $(1,6)$-step Hamiltonian since $D_1(P_{12}) \cup D_6(P_{12}) \cong 3 \times C_4$ with vertex sequences $0,6,7,1;2,8,9,3;4,10,11,5$. *However, P_{12} is $(2,3)$-step Hamiltonian with vertex sequence* $0,3,1,4,6,9,11,8,10,7,5,2,0$.

As shown in the proof of Theorem 3.2, the edges E of an (a,b)-step Hamiltonian cycle for $P_n = P_{2ab}$ must be composed of the union of sets of "parallel" length a edges with a similarly labeled collection of sets of "parallel" length b edges. Because each vertex in the set E is incident to one length a edge and one length b edge in E, the edges in E will constitute a set of cycles. If $a = 1, b = 2$, then P_4 is $(1,2)$-step Hamiltonian, with a cycle in the form C_4. Otherwise, in order to show that E is not composed of one (a,b)-step Hamiltonian cycle, we might simply find one example of a cycle C_4 in this collection.

When P_n is not (a,b)-step Hamiltonian the collection of these cycles exhibit interesting patterns, and this might be the subject of a separate investigation. For now we will establish conditions showing that in all cases P_n is not (a,b)-step Hamiltonian except for $a = 1$ and either $b = 2$ or b odd, and also for $a = 2$ and $b = 3$. We divide the cases into those in which $2a < b$, $3a/2 < b < 2a$, and $a < b < 3a/2$.

Theorem 3.3 *Suppose $gcd(a,b) = 1, n = 2ab, b > a \geq 1$, and the vertices of $P_n = P_{2ab}$ are labeled* $0,1,2,\ldots,n-1$, *with edges as described above.*

(i) If $a = 1$ then P_{2b} is $(1,b)$-step Hamiltonian if and only if b is odd, as shown in Example 3.1

(ii) If $a \geq 2$ with $gcd(a,b) > 1$, then P_n cannot be (a,b)-step Hamiltonian by Theorem 3.2.

(iii) If $a \geq 2$ and $2ma < b < (2m+1)a$ for some $m \geq 1$ then P_{2ab} is not (a,b)-step Hamiltonian.

(iv) If $a \geq 2$ and $(2m-1)a < b < 2ma$ for some $m \geq 2$ then P_{2ab} is not (a,b)-step Hamiltonian.

Proof. (iii) A copy of the 4-cycle C_4 among the edges is $(0,a,b,a+b)$. We have that $(0,a)$ and $(0,b)$ are edges in E. Since $a < b$ the length b edge incident to vertex a is $(a,a+b)$. Since $2ma < b < (2m+1)a$ the length a edge incident to vertex b is $(b,b+a)$.

(iv) A copy of the 4-cycle C_4 among the edges is $(a-1, 2a-1, a+b-1, 2a+b-1)$. We have that $(a-1, 2a-1)$ is an edge in E, and because $a-1 < b$, also $(a-1, a-1+b)$ is an edge in E. Since $(2m-1)a < b < 2ma$, we have $(2m-1)a + a - 1 < b + a - 1 < 2ma + a - 1$ so that $2ma - 1 < b + a - 1 < (2m+1)a - 1$. Therefore the length a edge at vertex $b + a - 1$ is $b + a - 1 + a = 2a + b - 1$. Since $m \geq 2$ we have $2a - 1 < 3a \leq (m-1)a < b$, so the length b edge incident to vertex $2a - 1$ is $(2a-1, 2a+b-1)$, and this completes the copy of C_4. $\square$

Example 3.2 *(iii): Let* $a = 4, b = 25, 2(3)4 < 25 < (2 \cdot 3 + 1)4$*,* $P_{(2)(4)(25)} = P_{200}$*,* C_4 *is* $(0, 4, 25, 29)$*. (iv): Let* $a = 4, b = 23, (2 \cdot 3 - 1)4 < 23 < (2 \cdot 3)4$*,* $P_{(2)(4)(23)} = P_{184}$*,* C_4 *is* $(3, 7, 26, 30)$*.*

The only remaining cases for paths are for $2 \leq a < b < 2a$. These are covered in the following.

Theorem 3.4 *If* $2 \leq a$ *and* $\frac{3}{2}a \leq b \leq 2a - 1$ *then* P_{2ab} *is not* (a, b)*-step Hamiltonian, except for* $a = 2, b = 3$*.*

Proof. Suppose $a \geq 3$, since the only other case is $a = 2, b = 3$, which does give the the (2,3)-step Hamiltonian path P_{12}, as mentioned previously. Note that we cannot have $b = \frac{3}{2}a$ since that is only possible if $a = 2m$ is even, in which case $gcd(a, b) = gcd(2m, 3m) = m > 1$. We next show that in the remaining cases $D_a(P_{2ab}) \cup D_b(P_{2ab})$ always contains a copy of C_4. Let $b = a + k$, where $\frac{a}{2} < k \leq a - 1$. We will try to find the smallest element x of a 4-cycle C_4 such that

$$a \leq x + a \leq b - 1$$

since we will want the b-edge at $x + a$ to be $(x + a, x + a + b)$. We also will want the a-edge at $x + b$ to be $(x + b, x + a + b)$, so we need

$$2a \leq x + b \leq 3a - 1$$

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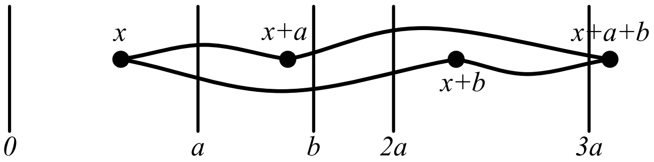


Figure 9: A 4-cycle in P_{2ab}

These inequalities simplify to

$$0 \leq x \leq b - a - 1$$

and

$$2a - b \leq x \leq 3a - b - 1$$

Because $b < 2a$ implies both that $0 < 2a - b$ and that $b - a - 1 < 3a - b - 1$ we have these boundaries for x:

$$2a - b \leq x \leq b - a - 1$$

or replacing b by $a+k$,

$$a-k \leq x \leq k-1$$

We can always find at least one value for x since the boundary values for x in this inequality, $a-k$ and $k-1$, are closest together when k is as small as possible, namely when $k = \lfloor \frac{a}{2} \rfloor + 1$. If $a = 2m$ is even, then for all values of k we have $a-k < \frac{a}{2} = m < k$, so we can choose $x = m-1$ which implies $a-k \leq m-1 = x \leq k-1$, as required. If $a = 2m+1$ is odd, then we can choose $x = m$ giving $a-k < \frac{a}{2} = m + \frac{1}{2} < k$, so we can choose $x = m$ again implying $a-k \leq m = x \leq k-1$. In both cases the inequality $a-k \leq x \leq k-1$ actually gives us a non-empty range of values for x that determine 4-cycles, implying that P_{2ab} is not (a,b)-step Hamiltonian. □

Example 3.3 *If $a = 7$ then the possible values for b within the constraints of this theorem are $b = 11, 12,$ or 13. Since $7 = 2(3)+1$ we can use $x = 3$ as the smallest value in a 4-cycle for each of these values of b, but the inequality for x also gives the following ranges for x:*

(i) $a = 7, b = 11, k = 4, x$ is in range $7-4 \leq x \leq 4-1$, so $x = 3$, the 4-cycle is $(3, 10, 21, 14)$.

(ii) $a = 7, b = 12, k = 5, x$ is in range $7-5 \leq x \leq 5-1$, so $x = 2, 3,$ or 4, example of a 4-cycle is $(2, 9, 21, 14)$.

(iii) $a = 7, b = 13, k = 6, x$ is in range $7-6 \leq x \leq 6-1$, so $x = 1, 2, 3, 4,$ or 5, example of a 4-cycle is $(5, 12, 25, 18)$.

Therefore in all of these examples P_{2ab} is not (a,b)-step Hamiltonian.

The remaining P_n cases are for $a < b < \frac{3}{2}a$. We have the following.

Theorem 3.5 *If $a < b < \frac{3}{2}a$, then P_{2ab} is not (a,b)-step Hamiltonian, if (i) a is odd and b is even, or (ii) a is even and b is odd.*

Proof. Example 3.1 shows that P_{12} is (2,3)-Hamiltonian.
(i) We show that $D_a(P_{2ab}) \cup D_b(P_{2ab})$ always contains a copy of C_4, $(ab, ab+a, ab+a-b, ab-b)$. Since ab is an even multiple of a, its a-edge at ab is $(ab, ab+a)$. Since $ab+a$ is between an odd multiple of b and an even multiple of b, its b-edge is $(ab+a, ab+a-b)$. Since ab is an odd multiple of b, its b-edge is $(ab, ab-b)$. Since

$$a(b-2) = ab-2a < ab-b < ab-a = a(b-1)$$

$ab-b$ is between the even multiple $a(b-2)$ of a and the odd multiple $a(b-1)$ of a, and so its a-edge must be $(ab-b, ab=b+a)$. See the two left-hand figures in Figure 10, including the example $a=7, b-10, p=2(7)(10)=140$, in which the horizontal segments are a-edges and the vertical segments are b-edges.

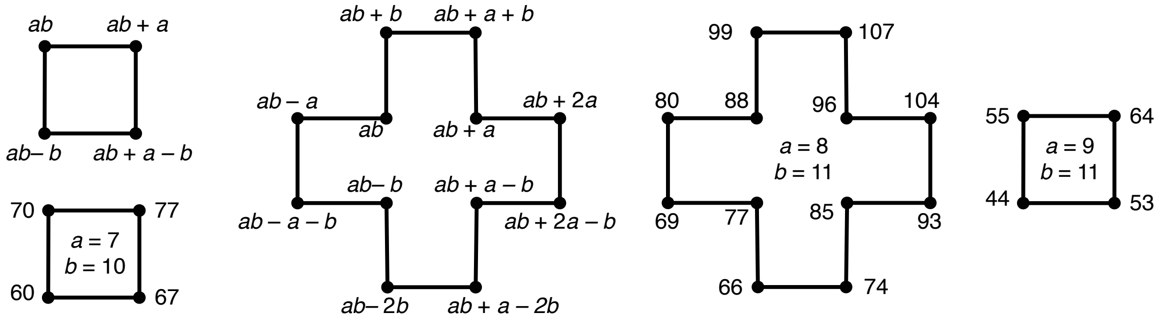


Figure 10: Examples of 4-cycles and 12-cycles

(ii) We show that $D_a(P_{2ab}) \cup D_b(P_{2ab})$ always contains a copy of C_{12} (see the two middle figures in Figure 10).

The following vertices have labels in intervals with minimum an even multiple of a and maximum one less than an odd multiple of a, so their a-edges are formed by adding a:
$ab-a=a(b-1)$, since $b-1$ is even.
$ab+a=a(b+1)$, since $b+1$ is even.
$ab+b$, in the interval $ab+a<ab+b<ab+2a$.
$ab-a-b$, in the interval $a(b-1)-2a<ab-a-b<a(b-2)$.
$ab+a-b$, in the interval $a(b+1)-2a=a(b-1)<ab+a-b<ab$.
$ab-2b$, in the interval $a(b-3)=ab-3a<ab-2b<ab-2a=a(b-2)$, since $a<b<\frac{3}{2}a$ or $2a<2b<3a$.
Similarly, the following vertices have labels in intervals with minimum an even multiple of b and maximum one less than an odd multiple of b, so their b-edges are formed by adding b:
ab, since a is even.
$ab-2b=b(a-2)$, since $a-2$ is even.
$ab+a$, in the interval $ab<ab+a<ab+b=b(a+1)$.
$ab-a-b$, in the interval $b(a-2)=ab-2b<ab-a-b<ab-b=b(a-1)$.
$ab+a-2b$, in the interval $b(a-2)=ab-2b<ab+a-2b<ab-b=b(a-1)$.
$ab+2a-b$, in the interval $ab=b(a-1)+b<b(a-1)+2a=ab+2a-b<ab+2b-b=b(a+1)$.

□

Theorem 3.6 *If $a < b < \frac{3}{2}a$, then P_{2ab} is not (a,b)-step Hamiltonian when a and b are both odd.*

Proof. We will show that we can always find a copy of C_4 with smallest element x in $D_a(P_{2ab}) \cup D_b(P_{2ab})$, where $x = 2ma + a - 1 = 2nb$, for some m and n. That is, x is $a-1$ more than an even multiple of a, and so its a-edge is $(a, x+a)$, and x equals an even multiple of b, so its b-edge is $(x, x+b)$. Then the b-edge at $x+a$ will be $(x+a, x+a+b)$, because $2nb < x+a < 2nb + b = (2n+1)b$. Also, the a-edge at $x+b$ will be $(x+b, x+b+a)$, because

$$\begin{gathered}(2m+2)a - 1 = 2m + a - 1 + a = x + a < x + b \\ < x + a + \frac{1}{2}a < x + 2a = 2ma + (a-1) + a = (2m+3)a - 1\end{gathered}$$

These will be true if $2ma + a = 1 + 2nb \equiv 1 \pmod{2b}$, that is if we can solve $(2m+1)a = ya \equiv 1 \pmod{2b}$, for some odd number y. This is always solvable because a is odd and relatively prime to both b and $2b$. Since $D_a(P_{2ab}) \cup D_b(P_{2ab})$ contains a copy of C_4 it cannot be (a,b)-step Hamiltonian. ☐

Example 3.4 *For $a = 9$ and $b = 11$, we solve $9y \equiv 1 \pmod{22}$, giving $y \equiv 5 \pmod{22}$, and $x = (9)(5) - 1 = 44 = (2)(22)$. The 4-cycle is then (44, 53, 64, 55), (see the right figure in Figure 10).*

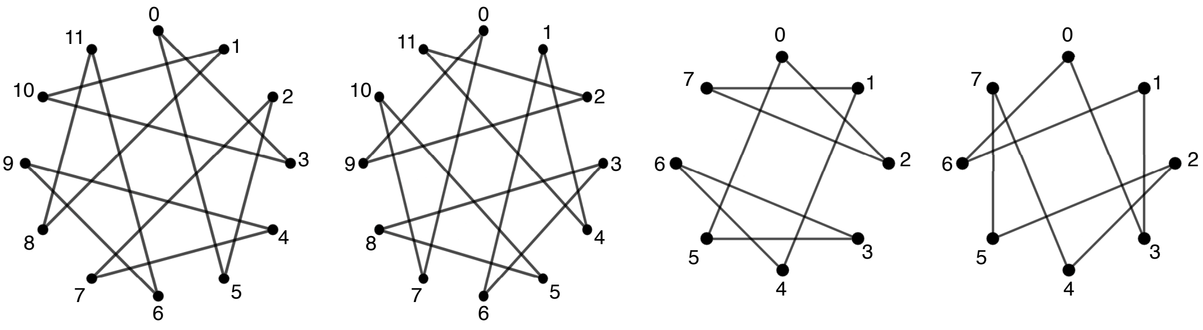
4 Cycles

Throughout this section we will use $[p] = \{0, 1, 2, ..., p-1\}$ as the vertex labels for the cycle C_p with edges $(i, i+1)$, with the understanding that vertices with labels congruent mod p are identified. $D_a(C_p) \cup D_b(C_p)$ is known as a circulant graph. Recently there has been interest in decompositions of circulant graphs into Hamiltonian cycles.

Theorem 4.1 *If a and b are both odd and relatively prime to p and C_p is (a,b)-step Hamiltonian then $D_a(C_p) \cup D_b(C_p)$ is decomposable into two (a,b)-step Hamiltonian cycles.*

Proof. Suppose $M_a(C_p)$ and $M_b(C_p)$ are perfect matchings subgraphs of $D_a(C_p)$ and $D_b(C_p)$ respectively, such that $M_a(C_p) \cup M_b(C_p)$ contains an induced cycle C_p, as described in Theorem 1.2. Let the edges of $M_a(C_p)$ be $(u_0, u_1), (u_2, u_3), ..., (u_{p-2}, u_{p-1})$ and the edges of $M_b(C_p)$ be $(v_0, v_1), (v_2, v_3), ..., (v_{p-2}, v_{p-1})$, where it is understood that $\{u_0, u_1, ..., u_{p-1}\} = \{v_0, v_1, ..., v_{p-1}\} = V(C_p)$. If we add an odd multiple $(2k+1)a$ of a to every vertex of $M_a(C_p)$ and reduce mod p, we will obtain the complementary perfect matching within $D_a(C_p)$ composed of edges $(u_0+(2k+1)a, u_1+(2k+1)a), (u_2+(2k+1)a, u_3+(2k+1)a), ..., (u_{p-2}+(2k+1)a, u_{p-1}+(2k+1)a)$. Similarly adding $(2k+1)b$ to each vertex of $M_b(C_p)$ produces the complementary perfect matching of $D_b(C_p)$. Let d equal the least common multiple of a and b, which must be odd. Adding d to each vertex of the edges of $M_a(C_p)$ and $M_b(C_p)$ produces a graph isomorphic to the cycle induced by $M_a(C_p)$ and $M_b(C_p)$ and thus to C_p.

Example 4.1 *Figure 11 shows a (3,5)-step Hamiltonian cycle for* C_{12} *along with its "rotated" complement obtained by adding the least common multiple of 3 and 5,* $15 \equiv 3 \pmod{12}$*, to every vertex. If* a *is even and* b *is odd the theorem may not be true, as the example on the right shows for the (2,3)-step Hamiltonian cycle for* C_8*, which has complementary edges in* $D_2(C_8)$ *and in* $D_3(C_8)$ *composing* $2C_4$*; all other (2,3)-step Hamiltonian cycles for* C_8 *are rotations or reflections of the one shown.*



(3,5)-step Hamiltonian cycle for C_{12} and complement

(2,3)-step Hamiltonian cycle for C_8 and complement

Figure 11: Hamiltonian cycles and complements for (3,5)-step C_{12} and (2,3)-step C_8

Theorem 4.2 *The cycle* C_p *is not* (a,b)*-step Hamiltonian if* $\frac{p}{gcd(a,p)}$ *or* $\frac{p}{gcd(b,p)}$ *is odd or if* a *or* $b > p/2$*. In particular, if* p *is odd, then* C_p *is not* (a,b)*-step Hamiltonian unless* $a = b = 1$*.*

Proof. If $\frac{p}{gcd(a,p)} = d$ is odd, then $D_a(C_p)$ is composed of $gcd(a,p)$ disjoint cycles C_d each of which cannot have a perfect matching, similarly for $\frac{p}{gcd(b,p)}$. If $a > p/2$, then $D_a(C_p)$ is composed of isolated vertices, similarly for b. The result holds by Theorem 1.2

Note that Theorem 4.2 guarantees that the only (a,b)-step Hamiltonian cycles C_p are those for which p is even. Furthermore, if either a or b is even, say a, and $p \equiv 2 \pmod 4$ then $\frac{p}{gcd(a,p)}$ is odd, contradicting the theorem. Therefore in this case p must be a multiple of 4, and similarly if a or b is divisible by a higher power of 2:

Corollary 4.3 *If a or b is even and C_p is (a,b)-step Hamiltonian then $p \equiv 0 \pmod 4$. If 2^k is the highest power of 2 that is a factor of either a or b and C_p is (a,b)-step Hamiltonian then 2^{k+1} is a factor of p.*

There are three possibilities for the parity of a and b, where $1 \leq a, b \leq \frac{p}{2}$ and $a \neq b$:

(i) a and b are both even. In this case no even label vertices and odd label vertices in $D_a(C_p) \cup D_b(C_p)$ are adjacent to each other, so $D_a(C_p) \cup D_b(C_p)$ is disconnected and C_p cannot be (a,b)-step Hamiltonian.

(ii) a and b are both odd.

(iii) One of a or b is odd and the other is even.

4.1 Cycles C_p with both a and b odd

We first examine case (ii) in which a and b are both odd. The simplest forms for an (a,b)-step Hamiltonian cycle are those in which to find the next vertex in the cycle we alternately add b and subtract a, or else those in which we alternately add b and add a; these correspond to sections (i) and (ii) respectively in the next theorem.

Theorem 4.4 *Suppose p is even and both a and b are odd, with $a < b \leq \frac{p}{2}$. Then C_p is (a,b)-step Hamiltonian if either of the following conditions are true:*

(i) $\frac{b-a}{2}$ is relatively prime to $\frac{p}{2}$.

(ii) $\frac{b+a}{2}$ is relatively prime to $\frac{p}{2}$

Proof. Let $m = \frac{b-a}{2}$.

(i) An (a,b)-step Hamiltonian cycle is given by the sequence $0, b, b-a, 2b-a, 2b-2a, 3b-2a, 3b-3a, \ldots, \frac{p}{2}b + (\frac{p}{2}-1)a$. Note that the elements of the sequence of the form $i(b-a)$ are even with respect to mod p, and the elements of the form $b + i(b-a)$ are odd with respect to mod p. These are all distinct, mod p, as the following three cases show.
If $i(b-a) \equiv j(b-a) \pmod p$ then $i\frac{b-a}{2} = im \equiv jm \pmod{\frac{p}{2}}$, and because $\gcd(m, \frac{p}{2}) = 1$, this implies that $i \equiv j \pmod{\frac{p}{2}}$. Because i and j take only the values $0, 1, 2, \ldots, \frac{p}{2}-1$, we must have $i = j$.
By a similar argument, $b + i(b-a) \equiv b + j(b-a) \pmod p$ also implies that $i = j$.
Because $b + i(b-a)$ and $j(b-a)$ are of oppositie parity, mod p, they also are distinct.

(ii) An (a,b)-step Hamiltonian cycle is given by the sequence $0, b, b+a, 2b+a, 2b+2a, 3b+2a, 3b+3a, \ldots, (\frac{p}{2})b+(\frac{p}{2}-1)a$. Similar arguments to those in part (i) show that if $i(b+a) \equiv j(b+a) \pmod{p}$ or if $(i+1)b+ia \equiv (j+1)b+ja \pmod{p}$ then $i=j$. As in case (i) the elements of form $(i+1)b+ia$ and those of the form $j(b+a)$ are of opposite parity with respect to mod p and so must be distinct as well.

□

Because for $b>1$ odd, $\frac{b-1}{2}$ must be relatively prime to b, we have

Corollary 4.5 *If* $a=1$ *and* $b=\frac{p}{2}$ *is odd then* C_p *is* $(1,b)$*-step Hamiltonian.*

Because $(b-(b-2c))/2 = c$ is relatively prime to $cn+1$ for any positive integers c and n, we have

Corollary 4.6 $C_p = C_{2cn+2}$ *is* $(b-2c,b)$*-step Hamiltonian for* b *odd and* n *a positive integer such that* $1<2c<b\leq cn+1=\frac{p}{2}$. *In particular, if* $c=1$ *then* C_p *is* $(b-2,b)$*-step Hamiltonian for any even value* p *such that* $3\leq b\leq \frac{p}{2}$.

Again let $m=\frac{b-a}{2}$ and let $k=\frac{b+a}{2}$.

Corollary 4.7 *If* a *and* b *are odd,* $a<b\leq\frac{p}{2}$, *and there is a positive integer* n *such that any of the following conditions are true then* C_p *is* (a,b)*-step Hamiltonian.*

(i) $\frac{p}{2}=mn\pm 1$
(ii) $\frac{p}{2}=kn\pm 1$
(iii)] $p=mn\pm 1$
(iv) $p=nk\pm 1$

Proof. For (i) m and $\frac{p}{2}=mn\pm1$ are relatively prime and similarly for (ii) k and $\frac{p}{2}=kn\pm1$ are relatively prime. For (iii) if $\frac{p}{2}$ and m share a common factor greater than 1 then we cannot also have $p=mn\pm1$, so (iii) implies that $\frac{p}{2}$ and $\frac{b-a}{2}$ are relatively prime, and similarly for (iv). □

Note that the conditions in this corollary are sufficient but not necessary. For example, if $p=32, a=3$ and $b=15$ then none of conditions (i) though (iv) are satisfied, yet $k=\frac{b+a}{2}=9$ is relatively prime to $\frac{p}{2}=16$ and so C_{32} is (3,15)-step Hamiltonian.

Example 4.2 *In Figure 12, we show how to construct* $(1,3)$*-step Hamiltonian tours for* C_6 *and* C_{10} *using the methods of Theorem 4.4.*

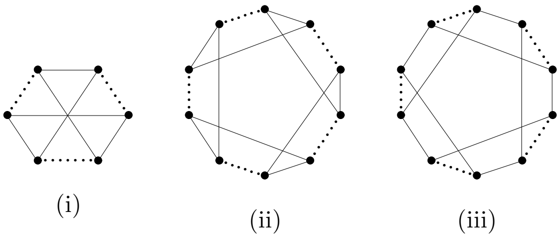


Figure 12: (1,3)-step Hamiltonian cycle(s) of C_6 and C_{10}

Example 4.3

C_{26} is $(3,11)$-step Hamiltonian since $\frac{11-3}{2} = 4$ is relatively prime to $\frac{26}{2} = 13$. The mod 26 vertex sequence is $0, 11, 8, 19, 16, 1, 24, 9, 6, 17, 14, 25, 22, 7, 4, 15, 12, 23, 20, 5, 2, 13, 10, 21, 18, 3$.

In [2] we have the condition that C_p is k-step Hamiltonian if and only if k is relatively prime to p, and [5] notes that if k is not relatively prime to p then we get disjoint cycles:

Theorem 4.8 *A cycle C_p is k-step Hamiltonian if and only if $p \geq 2k+1$ and $gcd(p,k) = 1$. If $gcd(p,k) = d > 1$ then $D_k(C_p)$ is composed of d disjoint copies of $C_{\frac{p}{d}}$.*

Theorem 4.9 *Suppose a and b are odd, p is even, $gcd(a,p) = gcd(b,p) = 1$, $d = gcd(\frac{b+a}{2}, \frac{p}{2}) > 1$, and $e = gcd(\frac{b-a}{2}, \frac{p}{2}) > 1$. Then C_p is not (a,b)-step Hamiltonian, and a two regular subgraph containing all the vertices of $D_a(C_p) \cup D_b(C_b)$ will be composed of either $dC_{\frac{p}{d}}$ or $eC_{\frac{p}{e}}$.*

Proof. By Theorem 4.8 $D_b(C_p)$ is k-step Hamiltonian and since p is even it has a matching subgraph which we may choose without loss of generality to have edges $M_b(C_p) = \{(0,b), (2b,3b), (4b,5b), ..., ((p-2)b, (p-1)b)\}$. We now have two choices for a matching subgraph $M_a(C_p)$, either $S_1 = \{(2x+b, 2x+b-a) | x = 0,1,2,...,p-1\}$ or $S_2 = \{(2x+b, 2x+b+a) | x = 0,1,2,...,p-1\}$. Suppose we choose S_1, consisting of edges connecting odd vertices $2x+b$ in C_p to even vertices $2x+b-a$, contract the $M_b(C_p)$ edges in $M_b(C_p) \cup S_1$, and create a mapping f from length 2 paths $\{(2x, 2x+b), (2x+b, 2x+b-a)\}$ in C_p to edges of $C_{\frac{p}{2}}$ such that $f(\{(2x, 2x+b), (2x+b, 2x+b-a)\}) = (x, x+\frac{b-a}{2})$, for $x = 0,1,2,...,p-1$. This results in a graph with $\frac{p}{2}$ vertices, $\frac{p}{2}$ edges E, and composed by Theorem 4.8 of e copies of $C_{\frac{p}{2e}}$. Then $f^{-1}(E)$ are the edges of e copies of $C_{\frac{p}{e}}$ in $D_a(C_p) \cup D_b(C_b)$. If we instead choose S_2 we similarly find a subgraph of $D_a(C_p) \cup D_b(C_b)$ composed of $dC_{\frac{p}{d}}$.

Note that as in Theorem 4.4, if either d or e is equal to 1, this theorem also gives us an (a,b)-step Hamiltonian cycle for C_p, although Theorem 4.4 does not require a and b to be relatively prime to p. ☐

The condition that a and b are relatively prime to p is necessary to the theorem, since for example C_{60} is actually (1,5)-step Hamiltonian, where $d = 3$ and $e = 2$, with one such (1,5)-step Hamiltonian cycle given by the vertex sequence
0,1,56,57,52,53,48,49,54,55,50,51,46,47,42,43,38,39,44,45,40,41,36,37,32,33,28,29,34,35,30,
31,26,27,22,23,18,19,24,25,20,21,16,17,12,13,8,9,14,15,10,11,6,7,2,3,58,59,4,5,0.

Theorem 4.10 *Suppose* $p \equiv 4 \pmod 8$, *gcd*$(a,p) =$ *gcd*$(b,p) = 1$, *and* $a + b = \frac{p}{2}$. *Then* C_p *is not* (a,b)*-step Hamiltonian, and a two regular subgraph containing all the vertices of* $D_a(C_p) \cup D_b(C_b)$ *will be composed of either* $2C_{\frac{p}{2}}$ *or* $\frac{p}{4}C_4$.

Proof. Let the vertices of C_p be labeled $0, 1, 2, \ldots, p-1$. Since p is a multiple of 4 and a and b are relatively prime to p, a and b must be odd. Since $\frac{p}{2} \equiv 2 \pmod 4$ either $a \equiv b \equiv 1 \pmod 4$ or $a \equiv b \equiv 3 \pmod 4$. There are two possibilities for a perfect matching of a-edges or of b-edges, and without loss of generality we may choose as a perfect matching of a-edges

$$\{(0,a), (2a,3a), (4a,5a), \ldots, ((p-2)a, (p-1)a)\} = R_1 \cup R_2,$$

where
$R_1 = \{(0,a), (4a,5a), \ldots, ((p-4)a, (p-3)a)\}$
$R_2 = \{(2a,3a), (6a,7a), \ldots, ((p-2)a, (p-1)a)\}$.
When $a \equiv 1 \pmod 4$, R_1 and R_2 are disjoint since their edges are of the form $(0,1)$ and $(2,3)$ (mod 4), respectively. When $a \equiv 3 \pmod 4$, R_1 and R_2 are still disjoint since their edges are of the form $(0,3)$ and $(2,1) \pmod 4$, respectively.
The two choices for perfect matchings of b-edges are of the form

(i) $S = \{(a, a+b), (3a, 3a+b), (5a, 5a+b), \ldots, ((p-1)a, (p-1)a+b)\}$.
(ii) $T = \{(a, a-b), (3a, 3a-b), (5a, 5a-b), \ldots, ((p-1)a, (p-1)a-b)\}$.

Note that because $(2i+1)a \equiv (2j+1)a \pmod p$ implies $i \equiv j \pmod{\frac{p}{2}}$ or $i = j$, the edges listed in S and also in T are not repeated, and the initial vertices of the edges in each run through every odd vertex, mod p. Furthermore, S and T are disjoint and $S \cup T = D_b(C_p)$.

(i) Suppose we choose the b-edges from S in case (i) along with the a-edges of $R_1 \cup R_2$ and let $x = 4ia$ for $i = 0, 1, 2, \ldots, \frac{p}{4} - 1$. Since $a + b = \frac{p}{2}$ a 2-regular subgraph containing all the vertices of $D_a(C_p) \cup D_b(C_b)$ will be composed of 4-cycles of the form

$$\{(x, x+a), (x+a, x+a+b), (x+a+b, x+2a+b), (x+2a+b, x+2a+2b)\}$$

$$\equiv \{(x,x+a),(x+a,x+\frac{p}{2}),(x+\frac{p}{2},x+\frac{p}{2}+a),(x+\frac{p}{2}+a,x)\} \pmod p$$

Here the a edge at $x+a+b$ must be$(x+a+b,x+2a+b)$ because $x+a+b$ is even, mod p. Similarly, $x+2a+b$ is odd, mod p, so to get the next vertex in the sequence we must add b to get the vertex $x+2a+2b$.

These 4-cycles are disjoint for the following reasons. The vertices alternate between even and odd elements, mod p. Since a is assumed relatively prime to p, if $4ia \equiv 4ja \pmod p$ then $i \equiv j \pmod{\frac{p}{4}}$, and this implies that $i=j$ since i and j can only take the $\frac{p}{4}$ values $0,1,\ldots \frac{p}{4}-1$. The other pairs of even vertices in these 4-cycles are distinct, since $4ia \equiv 4ja+\frac{p}{2} \pmod p$ would imply that $0 \equiv 2 \pmod 4$, because $\frac{p}{2} \equiv 2 \pmod 4$. Similar calculations show that even vertices of the form $x+a+b$ and the odd vertices in these 4-cycles are distinct. There will be $\frac{p}{4}$ such 4-cycles.

(ii) Next we choose the b-edges from T in case (ii) along with the a-edges of $R_1 \cup R_2$. The sequence of $\frac{p}{2}$ vertices

$$0,a,a-b,2a-b,2a-2b,3a-2b,\ldots,(\frac{p}{4}-1)(a-b),(\frac{p}{4}-1)(a-b)+a,\frac{p}{4}(a-b)\equiv 0$$

form a $\frac{p}{2}$-cycle for the following reasons. These vertices are alternately even and odd, mod p, and the edges are alternately of length a and b. Suppose that two of the even vertices in this set are congruent, mod p; that is, for some i and j, $i(a-b) \equiv j(a-b) \pmod p$. Because a and b are both congruent to either 1 or to 3, mod 4, $a-b$ must be divisible by 4, so we have $(i-j)\frac{a-b}{4} \equiv 0 \pmod{\frac{p}{4}}$. However, $\frac{p}{4}$ must be odd because $p \equiv 4 \pmod 8$, so any factor d common to $\frac{a-b}{4}$ and $\frac{p}{4}$ must be odd, and would also be an odd factor common to both $a-b$ and $2 \cdot \frac{p}{4} = \frac{p}{2} = a+b$ and thus also to $(a-b)+(a+b)=2a$. Because a and p are relatively prime, we must have common factor $d=1$ and this implies $\frac{a-b}{4}$ and $\frac{p}{4}$ are relatively prime, so the congruence $(i-j)\frac{a-b}{4} \equiv 0 \pmod{\frac{p}{4}}$ reduces to $i-j \equiv 0 \pmod{\frac{p}{4}}$, and therefore $i=j$, given that $0 \le i,j \le \frac{p}{4}-1$. Note that the last edge in this cycle sequence has terminal vertex $\frac{p}{4}(a-b) \equiv p\frac{a-b}{4} \equiv 0 \pmod p$. Similar calculations show that the odd vertices, mod p, in this sequence are also distinct.

Lastly we note that if $a \equiv b \equiv 1 \pmod 4$ then the vertices in the sequence above are all congruent to 0 or 1, mod 4, and the a-edges are the edges in R_1. Similarly, if $a \equiv b \equiv 3 \pmod 4$ then the vertices in the sequence are all congruent to 0 or 3, mod 4, and the a-edges again are the edges in R_1. In either case the a-edges in R_2 are part of a second length $\frac{p}{2}$ cycle with vertices in the sequence

$$2,2+a,2+a-b,2+2a-b,2+2a-2b,2+3a-2b,\ldots,2+(\frac{p}{4}-1)(a-b),2+(\frac{p}{4}-1)(a-b)+a$$

If $a \equiv b \equiv 1 \pmod 4$ then the vertices in the sequence are all congruent to 2 or 3, mod 4. If $a \equiv b \equiv 3 \pmod 4$ then the vertices are all congruent to 1 or 2, mod 4. In both cases the two $\frac{p}{2}$-cycles are disjoint, and have edges that are alternately of length a and b.

In both case (i) and (ii) C_p is not (a, b)-step Hamiltonian. $\square$

If we let $p = mkn \pm 1$, where $m = \frac{b-a}{2}$ and $k = \frac{b+a}{2}$ as before, then for any $n > 1$, we see that Corollary 4.7 implies the following:

Theorem 4.11 *For each a and b both odd and distinct there are an infinite number of values p, with $a < b \leq \frac{p}{2}$, such that C_p is (a, b)-step Hamiltonian.*

For a and b both odd, if the the conditions above are not satisfied, then C_p may or may not be (a, b)-step Hamiltonian. That is, suppose $\gcd(m, \frac{p}{2}) > 1$, $\gcd(k, \frac{p}{2}) > 1$, either $\gcd(a, p) > 1$ or $\gcd(b, p) > 1$, and $p \equiv 0 \pmod 4$. A Mathematica search shows that for the 440 values of $p \leq 200$ and $a, b < \frac{p}{2}$ satisfying these conditions, in 394 cases C_p is (a, b)-step Hamiltonian, and in 46 cases it is not. The smallest value of p for which $p \equiv 2 \pmod 4$ and these conditions are met is $p = 210$, and for all 72 such triples $\{a, b, p\}$, C_{210} is (a, b)-step Hamiltonian.

4.2 Cycles C_p with a even and b odd

Theorem 4.12 *Suppose b is odd and relatively prime to p, a is even, and $4 \leq a < \frac{p}{2}$. Then C_p is (a, b)-step Hamiltonian if and only if $p \equiv 0 \pmod 4$ and $\gcd(p/2, a/2) = 1$.*
If b is odd and not relatively prime to p, but $p \equiv 0 \pmod 4$, a is even and $\gcd(p/2, a/2) = 1$, then C_p is also (a, b)-step Hamiltonian.

Proof. Suppose first that $p \equiv 0 \pmod 4$ and $\gcd(p/2, a/2) = 1$. An (a, b)-step Hamiltonian cycle is $0, b, a + b, a, 2a, 2a + b, 3a + b, 3a, 4a, 4a + b, 5a + b, \ldots, (\frac{p}{2} - 1)a + b, (\frac{p}{2} - 1)a, 0$. Here the even elements in the vertex list $0 \cdot a, 1 \cdot a, 2a, \ldots, (\frac{p}{2} - 1)a$ are distinct, $\pmod p$, since if $ia \equiv ja \pmod p$, then $i \equiv j \pmod{\frac{p}{2}}$, because $\gcd(a, p) = 2$. This implies also that $i = j$, and the elements in the list $0 \cdot a, 1 \cdot a, 2a, \ldots, (\frac{p}{2} - 1)a$ are members of the set of residues $\{0, 2, 4, \ldots, p - 2\}$, $\pmod p$. Similarly, the odd elements $0 \cdot a + b, 1 \cdot a + b, 2a + b, \ldots, (\frac{p}{2} - 1)a + b$ are also distinct, $\pmod p$, and they are the elements of the set of residues $\{1, 3, 5, \ldots, p - 1\}$, $\pmod p$. Note that we cannot have $a = \frac{p}{2}$ since then $\gcd(a, p) \neq 2$, unless $p = 4$, which is only possible if b=1 and a=2, in which case the theorem is satisfied.

Now suppose that C_p is (a, b)-step Hamiltonian. Since a is even we must have $p \equiv 0 \pmod 4$. As in the proof of Theorem 4.9 we may without loss of generality choose to include length b edges $M_b(C_p) = \{(0, b), (2b, 3b), (4b, 5b), ..., ((p-2)b, (p-1)b)\}$. These are distinct because b is relatively prime to p. Note also that each edge in this perfect matching is of the form $(x, x+b)$, where x is even and $x+b$ is odd. Beginning with edge $(0, b)$, the length a edge incident to vertex b may be either $(b, b+a)$ or $(b, b-a)$; it may be seen that each choice leads to equivalent results, so without loss of generality suppose we choose $(b, b+a)$. Because $b+a$ is odd, the next edge in this sequence must be $(b+a, a)$. The edge $(a, 0)$ would form a 4-cycle, so in order to attempt to form an (a, b)-step Hamiltonian cycle, the next length a edge must be $(a, 2a)$. This pattern of four edges now repeats leading from vertex $2a$ to vertex $4a$, and for all xa such that x is in $\{0, 2, 4, ..., p-2\}$. A cycle is formed for the smallest even value of y such that $ya \equiv 0 \pmod p$, or $y\frac{a}{2} \equiv 0 \pmod{\frac{p}{2}}$. This value of y is given by $y = \frac{p}{d}$ where $d = gcd(\frac{p}{2}, \frac{a}{2})$, producing d copies of $C_{\frac{p}{d}}$. Only if $d = 1$ do we have an (a, b)-step Hamiltonian cycle.

If b is odd and not relatively prime to p, $p \equiv 0 \pmod 4$, a is even and $gcd(p/2, a/2) = 1$, then the same construction as that above gives an (a, b)-step Hamiltonian cycle; in this case the perfect matching of b edges will again be all edges of the form $(x, x+b)$ for all even x. □

Corollary 4.13 *If b is odd, $p > 4$, and $a = \frac{p}{2}$ is even, then C_p is not (a, b)-step Hamiltonian.*

Proof. Let $f = \frac{p}{gcd(b,p)}$. Then $D_b(C_p)$ is composed of $\gcd(b, p)$ copies of disjoint f–cycles. Either $(0, b)$ or $(0, -b)$ will need to be an edge of the Hamiltonian p–cycle. Without loss of generality we may choose the $(0, b)$ edge, which implies that $(a, 0)$ which is identical to $(-a, 0)$ and $(b, b+a)$ which is identical to $(b, b-a)$ will also be edges. The other edges of the matching $M_b(C_f)$ containing the edge $(0, b)$ will all be of the form $(x, b+x)$ where x is even. Therefore $(a, b+a)$ will also be an edge creating the 4–cycle $(0, b, b+a, a)$. So C_p cannot be (a, b)-step Hamiltonian unless $p = 4$ (in which case we will have $b = 1$ and $a = 2$). Similarly if the b edge at 0 is $(0, -b)$.

Corollary 4.14 *All even cycles C_p are $(2, b)$-step Hamiltonian for odd $b < \frac{p}{2}$ if and only if $p \equiv 0 \pmod 4$.*

See the examples in Figure 13 for $p = 12$. □

Corollary 4.15 *Let a and p be even with $4 \leq a < \frac{p}{2}$. The cycle C_p is $(1, a)$-step Hamiltonian if and only if $p \equiv 0 \pmod 4$ and $gcd(p/2, a/2) = 1$.*

Example 4.4 *A $(1, 6)$-step Hamiltonian tour for C_{16} is given by*

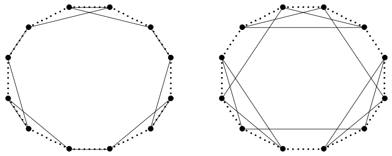


Figure 13: (1,2)- and (2,3)-step Hamiltonian tours for C_{12}

$$0, 1, 7, 6, 12, 13, 3, 2, 8, 9, 15, 14, 4, 5, 11, 10, 0.$$

A Mathematica search has demonstrated the following conjecture for the 152 triples $\{a, b, p\}$ fitting the conditions and such that $p \leq 100$.

Conjecture 4.1 *If b is odd, gcd*$(b, p) > 1$, *a is even, gcd*$(\frac{a}{2}, \frac{p}{2}) > 1$, *gcd*$(a, b, p) = 1$, *and* $a, b < \frac{p}{2}$ *then* C_p *is not* (a, b)*-step Hamiltonian.*

Theorem 4.16 *All even cycles* C_p *are not* $(3, 4)$*-step Hamiltonian.*

Proof. If $p \not\equiv 0 \pmod 8$, then $D_4(C_p)$ has odd cycles as components, so by Corollary 1.3, C_p is not $(3, 4)$-step Hamiltonian. If $p \equiv 0 \pmod 8$, the union of the spanning matching subgraph of $D_3(C_p)$ and of $D_4(C_p)$ induces a disconnected graph composed of two copies of $C_{p/2}$. We show this using two cases: (I) p not divisible by 3, (II) p divisible by 3. Let $p = 8k$, and label the vertices of C_p as $0, 1, 2, \ldots, 8k - 1$.

(I) By Theorem 4.12 C_p cannot be (3,4)-step Hamiltonian; however, we will give more details which will be useful in part (ii). In this case $D_3(C_p)$ is isomorphic to C_p and $D_4(C_p)$ is composed of four disjoint copies of C_{2k}. When trying to find a Hamiltonian cycle which alternates between the edges of $D_4(C_p)$ and $D_3(C_p)$, we may choose without loss of generality the perfect matching in $D_3(C_p)$ with edges $(0, 3), (6, 9), \ldots, (8k - 4, 8k - 1), (2, 5), (8, 11), (14, 17), \ldots, (8k - 8, 8k - 5), (8k - 2, 1), (4, 7), (10, 13), \ldots, (8k - 12, 8k - 9), (8k - 6, 8k - 3)$, as shown by the darker vertical edges in Figure 14. Again without loss of generality we may choose either edges in $D_4(C_p)$ that include either of the horizontal edges $(0, 4)$ or $(3, 7)$, but not both. If we choose $(0, 4)$ then we must also choose $(7, 11), (8, 12), \ldots, (8k - 8, 8k - 4), (8k - 1, 3)$, which creates a cycle isomorphic to C_{4k}, as shown in Figure 14. Similarly the edges $(6, 9), (10, 13), \ldots$ are part of a cycle isomorphic to C_{4k}, also shown in Figure 14. Therefore $D_3(C_{8k}) \cup D_4(C_{8k})$ induces a disconnected graph composed of two copies of C_{4k}, and C_{8k} cannot be (a, b)-step Hamiltonian.

(II) In this case $p = 24k$, since p is divisible by both 8 and 3. $D_3(C_p)$ is composed of three disjoint copies of C_{8k}, as shown by all the horizontal edges in Figure 15 (a) for the example $p = 24$. $D_4(C_p)$ is similarly composed of four copies of C_{6k}, shown by the vertical edges. When

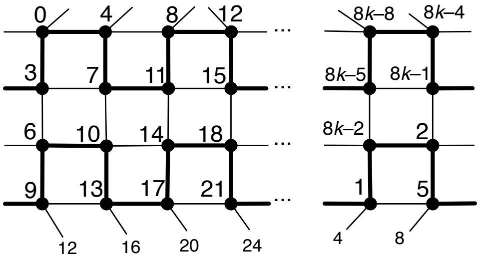


Figure 14: Case (I) in which n is not divisible by 3.

attempting to find a $(3, 4)$-step Hamiltonian cycle we have two choices; we might choose a perfect matching from $D_3(C_p)$ in which each matched edge has labels 4 more, mod p, than one of the other matched edges, as in the list of edges (0,3), (4,7), (8,11), (6,9), (10,13), (14,17),…, as shown by the darker horizontal edges for $p = 24$ in Figure 15(a). Otherwise we must choose edges from two cycles of the matching in which matched edges differ in edge labels by 4, and one cycle with edges complementary to those, as shown without loss of generality by the darker horizontal edges in Figure 15 (b). In the first case, when next choosing the alternate vertical edges from $D_4(C_n)$ we may choose either (0,4) or (3,7); Figure 15(a) shows the choice (0,4) which forces a copy of C_{12} within $D_3(C_{8k}) \cup D_4(C_{8k})$, for this example in which $p = 8k = 24$, namely (0,4,7,11,8,12,15,19,16,20,23,3), while the choice (3,7) also leads to a copy of C_{12}. There is a similar result for the other six horizontal edges, (6,9), (10,13), ... which also leads to selection of 12 edges forming a copy of C_{12}. Therefore, in this case all possible choices of the vertical edges from $D_4(C_p)$ create two disjoint copies of C_{12k}. In the second case, shown in Figure 15 (b) there are two vertical edge choices to be made for each set of 12 edges. For example, the figure shows edge (3,7) chosen rather than (04), which then leads to a choice between (1,5) and (5,9); in this diagram (1,5) is chosen, producing the 12-cycle (0,3,7,4,8,5,1,22,18,21,17,20). Again each such choice leads to two disjoint copies of C_{12}. These results apply in a similar manner for all $n = 24k$; in these cases our diagram will again have 3 rows of horizontal edges, but $8k$ columns of vertical edges. $\square$

Theorem 4.17 *Let* $a = 2^c g$ *and* $p = 2^d h$*, where* g *and* h *are odd,* $d > c > 1$*,* $gcd(p, a) = 2^c$*,* $4 \le a < \frac{p}{2}$ *and* $gcd(p, b) = 1$*. Also let* $e = 2^{c-1}$ *and* $f = \frac{p}{2^{c+1}}$*. Then* C_p *is not* (a, b)*-step Hamiltonian, and any 2-regular subgraph containing all the vertices of* $D_a(C_p) \cup D_b(C_b)$ *will be composed of one of* $\{(e - i)C_{4f} \cup ifC_4\}_{i=0}^{e}$*.*

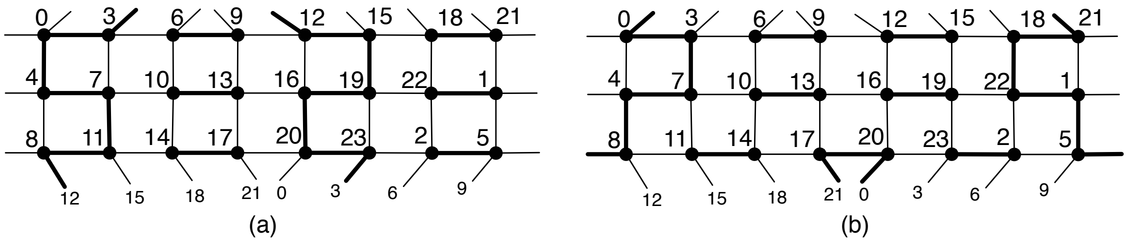


Figure 15: Case (II) in which n is divisible by 3.

Proof. Partition the vertex labels of C_p as follows:

$$S = \{0, 1, 2, ..., p-1\} = \bigcup_{i=0}^{2^c-1} S_i, \quad S_i = \{xb \equiv i \pmod{2^c} \mid x \in \{0, 1, 2, ..., p-1\}\}$$

The sets S_i are disjoint and each has $\frac{p}{2^c} = 2f$ elements. Without loss of generality we may choose b-edges $(0, b), (2b, 3b), ..., ((\frac{p}{2} - 2)b, (\frac{p}{2} - 1)b)$. These edges create a perfect matching between S_i and S_{i+1}, for each $i = 0, 2, 4, ..., 2^c - 2$. For each even value i we may without loss of generality choose as a-edges with vertices in S_i the edges $(ib, ib+a), (ib+2a, ib+3a), ..., (ib+(\frac{p}{2^c}-2)a, ib+(\frac{p}{2^c} - 1)a)$. For each corresponding odd value $i + 1$ we then have two choices for the a-edges within S_{i+1}, either those including $((i+1)b, (i+1)b+a)$ or else $((i+1)b+a, (i+1)b+2a)$. The former produces with vertices in $S_i \cup S_{i+1}$ f copies of C_4. The latter produces one copy of C_{4f} from the $2 \cdot 2f$ vertices in $S_i \cup S_{i+1}$. There are e pairs S_i and S_{i+1}, each of which may produce either C_{4f} or fC_4. See the following example. □

Example 4.5 *Figure 16 shows* $C_{40}, b = 7, a = 12$ *and the three possible 2-regular graphs induced as in Theorem 4.17. In this example* $c = 2, e = 2^{c-1} = 2, f = \frac{p}{2^{c+1}} = \frac{40}{2^{2+1}} = 5$, *and possible 2-regular subgraphs are* $\{(2-i)C_{\frac{40}{2}} \cup 5iC_4\}_{i=0}^{2} = \{2C_{20}, C_{20} \cup 5C_4, 10C_4\}$.

The above theorems do not completely characterize which C_p are (a, b)-step Hamiltonian. For $p \leq 200$ there are 126,225 triples $\{a, b, p\}$ with $a, b \leq \frac{p}{2}$. The theorems in this paper cover 98.6 % of these triples, demonstrating that there are 70,607 cases which are (a, b)-step Hamiltonian, 53,846 that are not (a, b)-step Hamiltonian, and 1,772 that are not decided by these theorems.

5 Turtle Shell Graphs

For $s \geq r \geq 1$ and $n = 2r + 2s$, let C_n be a cycle with $E(C_n) = \{a_i a_{i+1}, b_j b_{j+1}, c_k c_{k+1}, d_l d_{l+1} \mid 1 \leq i, k \leq r-1, 1 \leq j, l \leq s-1\} \cup \{a_r b_1, b_s c_1, c_r d_1, d_s a_1\}$. Denote by $TS(r, s)$ the turtle shell

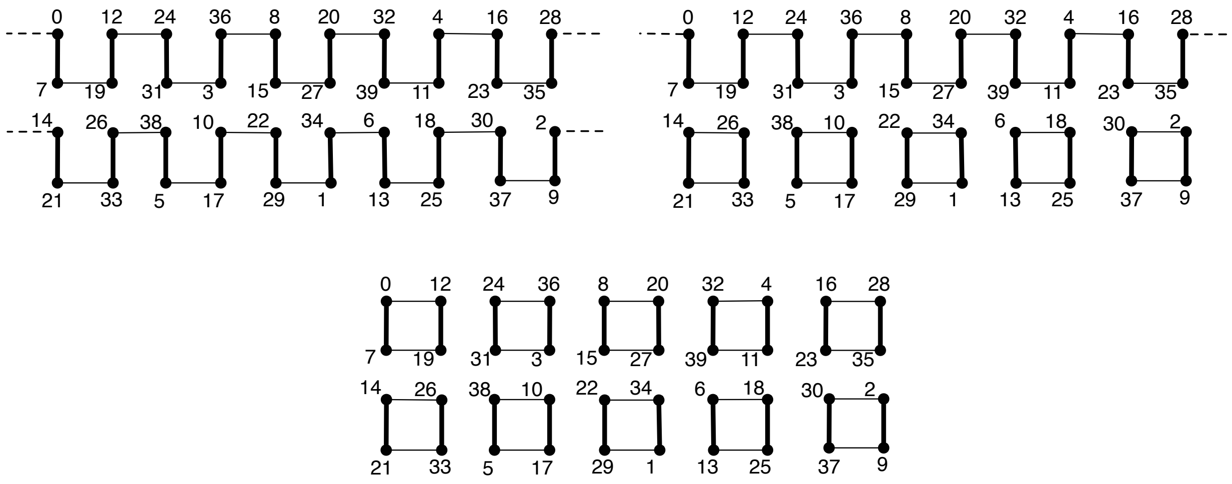
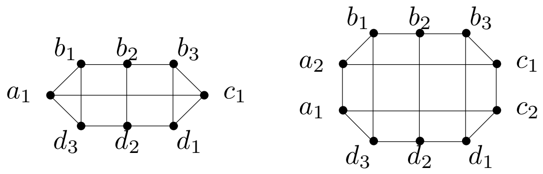


Figure 16: $C_{40}, a = 7, b = 12$ induces $2C_{20}$ or $C_{20} \cup 5C_4$ or $10C_4$.



$TS(1,3)$ $TS(2,3)$

Figure 17: Turtle shell graphs $TS(1,3)$ and $TS(2,3)$

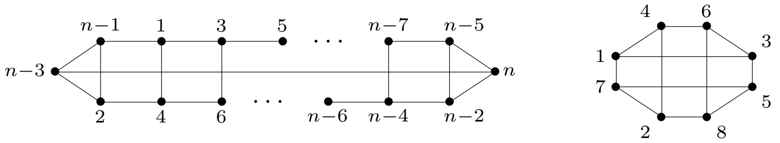
graphs obtained from C_n by adding the edges $\{a_i c_{r-1-i}, b_j d_{s-1-j}\,|1 \leq i \leq r, 1 \leq j \leq s\}$ as shown in Figure 17.

Theorem 5.1 *For $s \geq r \geq 1$, the turtle shell graph $TS(r,s)$ is (i) (1,2)-step Hamiltonian except $r = s = 1$; and (ii) (2,3)-step Hamiltonian except $r = 1, 1 \leq s \leq 3$.*

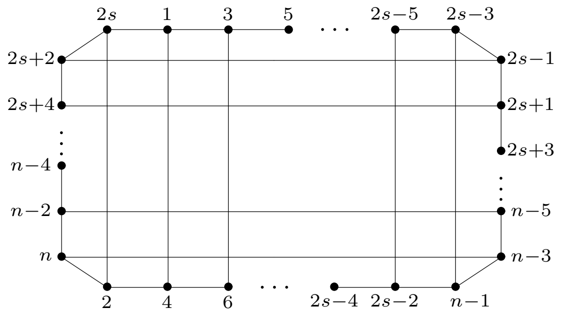
Proof. (i) Clearly, $TS(1,1)$ is not (1,2)-step Hamiltonian. For all other values of r, s, a (1,2)-step Hamiltonian tour is given by $a_1c_ra_2c_{r-1}a_3 \cdots a_rc_1b_1d_sb_2d_{s-1}b_3 \cdots b_sd_1a_1$.

(ii) It is routine to check that $TS(r,s)$ is not (2,3)-step Hamiltonian for $r = 1, 1 \leq s \leq 3$. For $r = s = 2$, or $r = 1, s \geq 4$, or $r \geq 2, s \geq 3$, a (2,3)-step Hamiltonian tour is given by the labeling in Figure 18. □

Problem 5.1 *Prove or disprove: For $a \geq 3, r \geq a, s \geq 2a-2$, the turtle shell graph $TS(r,s)$ is $(a, a+1)$-step Hamiltonian.*



$TS(1,s)$, $n = 2s+2$ $TS(2,2)$



$TS(r,s), n = 2r+2s, s \geq r \geq 2, s+r \geq 5$

Figure 18: (2,3)-step Hamiltonian tour for $TS(r,s)$ ($r = s = 2$, or $r = 1, s \geq 4$, or $r \geq 2, s \geq 3$)

6 Cartesian Products

The Cartesian product of graphs G and H is the graph $G \times H$ with vertices $V(G) \times V(H)$. Vertices (u,v) and (u',v') are adjacent in $G \times H$ if and only if
$u = v$ and u' is adjacent to v' in H, or if
$u' = v'$ and u is adjacent to v in G.

Lemma 6.1 *If the distance between two vertices u and v in G is a then for any vertex y in H the distance between (u,y) and (v,y) in $G \times H$ is also a. Similarly, if $d(u,v) = a$ for vertices u and v in H, then for any vertex x in G, $d((x,u),(x,v))$ in $G \times H$ is also a.*

Proof. Suppose $d(u,v) = a$ for vertices u and v in $V(G)$, and for $y \in V(H)$ let $G_y = \{(w,y)|w \in V(G)\}$. Let $P = (u,u_1),(u_1,u_2),...,(u_{n-1},v)$ be a length n path from (u,y) to (v,y) in $G \times H$. Contracting all edges in P that are not in G_y results in a subgraph of $G \times H$ contained within G_y, which implies that $n \geq a$, since G_y is isomorphic to G. □

Theorem 6.2 *If G and H are both (a,b)-step Hamiltonian then $G \times H$ is (a,b)-step Hamiltonian.*

The labeling in the following proof is illustrated in Example 6.1 and Figure 19.

Proof. Suppose G is of order m and H is of order n, both of which must be even. Let (a, b)-step Hamiltonian cycles in G and H be $u_1, u_2, u_3, ..., u_m, u_1$ and $v_1, v_2, v_3, ..., v_n, v_1$ respectively. The edges in one (a, b)-step Hamiltonian cycle for $G \times H$ are the following (not in this order):
In H_{u_1}: $((u_1, v_1), (u_1, v_2)), ((u_1, v_2), (u_1, v_3)), ((u_1, v_3), (u_1, v_4))..., ((u_1, v_{n-1}), (u_1, v_n))$,
In H_{u_2}: $((u_2, v_1), (u_2, v_2)), ((u_2, v_3), (u_2, v_4)), ((u_2, v_5), (u_2, v_6))..., ((u_2, v_{n-1}), (u_2, v_n))$,
In H_{u_m}: $((u_m, v_2), (u_m, v_3)), ((u_m, v_4), (u_m, v_5)), ((u_m, v_6), (u_m, v_7))..., ((u_m, v_{n-2}), (u_m, v_{n-1}))$
All edges in each G_x except
$((u_1, v_1), (u_2, v_1)), ((u_1, v_2), (u_2, v_2)), ((u_1, v_3), (u_2, v_3))..., ((u_1, v_n), (u_2, v_n))$, and
$((u_1, v_2), (u_m, v_2)), ((u_1, v_3), (u_m, v_3)), ((u_1, v_4), (u_m, v_4)), ..., ((u_1, v_{n-1}), (u_m, v_{n-1}))$. ☐

Example 6.1 *Figure 19 shows* $(1, 3)$*-step Hamiltonian cycles in* C_8 *and* C_6*, and the associated* $(1, 3)$*-step Hamiltonian cycle in* $C_8 \times C_6$ *described in the proof of Theorem 6.2. Note that the diagram does not include all edges in* $D_1(C_8 \times C_6) \cup D_3(C_8 \times C_6)$*, only those that are either in* $(C_8)_y$ *for some* $y \in V(C_6)$ *or in* $(C_6)_x$ *for some* $x \in V(C_8)$*. The broken edges represent distance 1 edges and the solid edges represent distance 3 edges. The thick edges are the edges in a* $(1, 3)$*-step Hamiltonian cycle for* $C_8 \times C_6$*.*
Because no edges of the form $((x, 3), (x, 0))$ *for every* $x \in V(C_8)$ *are included in this* $(1, 3)$*-step Hamiltonian cycle for* $C_8 \times C_6$*, the same Hamiltonian cycle will work for* $C_8 \times P_6$*, where we use the first 5 edges in the (1,3)-step Hamiltonian cycle for* C_6 *given by the vertex sequence 0,1,4,5,2,3 for the (1,3)-step traceable path* P_6*.*

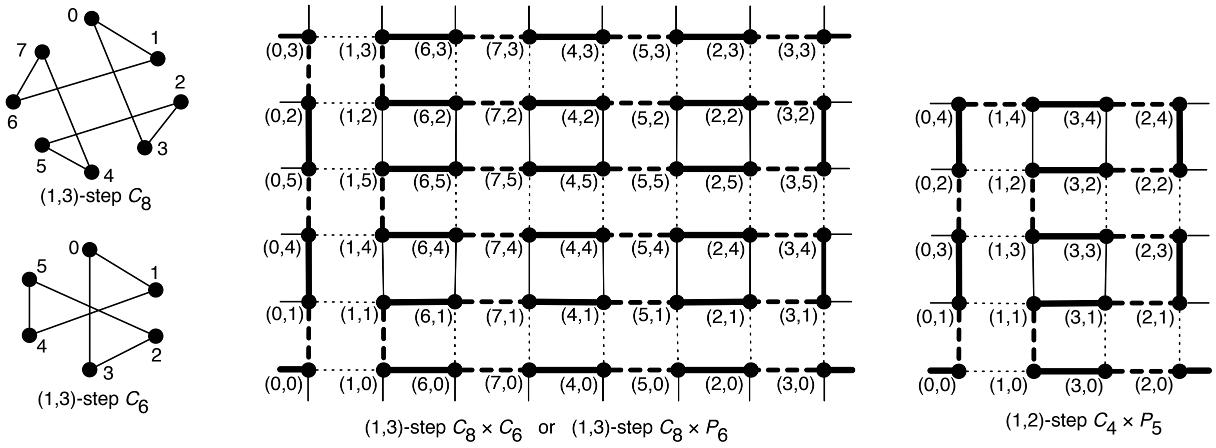


Figure 19: $(1, 3)$-step $C_8 \times$ $(1, 3)$-step C_6, $(1, 2)$-step $C_4\times$ $(1, 2)$- step traceable P_5

Corollary 6.3 *If* G *is* (a, b)*-step Hamiltonian and* H *is* (a, b)*-step traceable, then* $G \times H$ *is* (a, b)*-step Hamiltonian.*

Proof. Suppose n is the order of H and m, which must be even, is the order of G. As pointed out in example 6.1, the (a,b)-step Hamiltonian cycle used in the proof of Theorem 6.2 does not include any edges of the form $((x,v_n),(x,v_1))$ for any vertex $x \in V(G)$. Therefore the same (a,b)-step Hamiltonian cycle is also an (a,b)-step Hamiltonian cycle for $G \times H$, as long as n is even. If n is odd, then we remove edge $((u_1,v_n),(u_m,v_n))$ and replace it with edge $((u_1,v_n),(u_2,v_n))$, as in the example of the (1,2)-step Hamiltonian cycle for $C_4 \times P_5$ shown in Figure 19. □

Theorem 6.4 *The cylinder graph* $C_n \times P_m$ *is* $(1,2)$*-step Hamiltonian for (i)* $n \geq 3$ *and even* $m \geq 2$*; and (ii)* $n \equiv 0 \pmod 4$*, an,d odd* $m \geq 3$*.*

Proof. In Figure 20, we give a $(1,2)$-step Hamiltonian tour for $C_5 \times P_4$ (and $C_8 \times P_3$) obtained from a (1,2)-step Hamiltonian tour for $C_3 \times P_2$ (and $C_4 \times P_3$) with the same pattern may be obtained for other n and even m. □

Figure 20: A (1,2)-step Hamiltonian tour for $C_5 \times P_4$ and for $C_8 \times P_3$

Theorem 6.5 *The cylinder graph* $C_n \times P_m$ *is* $(2,3)$*-step Hamiltonian for*

(i) $n = 3, m \equiv 0 \pmod 4$*,*

(ii) $n \geq 4$*, and even* $m \geq 2$*,*

(iii) $n \equiv 0 \pmod 4$*, and odd* $m \geq 3$*.*

Proof. (i) For C_3, the distance between any two vertices is 1. A $(2,3)$-step Hamiltonian tour for $C_3 \times P_{12}$ obtained from a $(2,3)$-step Hamiltonian tour for $C_3 \times P_4$ is given in Figure 21 with the same pattern may be extended to other $m \equiv 0 \pmod 4$ larger than 12.

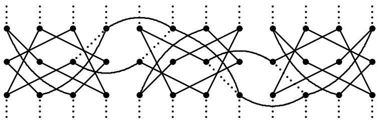


Figure 21: A (2,3)-step Hamiltonian tour for $C_3 \times P_{12}$

(ii) In Figure 22, we give a $(2,3)$-step Hamiltonian tour for $C_4 \times P_6$ obtained from a $(2,3)$-step Hamiltonian tour for $C_4 \times P_2$ that can be extended to other $n \geq 5$ and even $m \geq 4$.

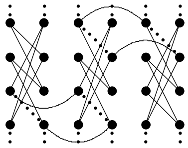


Figure 22: (2,3)-step Hamiltonian tour for $C_4 \times P_6$

(iii) In Figure 23, we give a $(2,3)$-step Hamiltonian tour for $C_8 \times P_5$ obtained from a $(2,3)$-step Hamiltonian tour for $C_3 \times P_4$ and $C_2 \times P_4$ that can be extended to other $n \equiv 0 \pmod 4$ and odd $m \geq 7$. $\square$

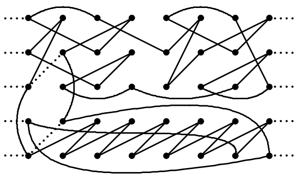


Figure 23: (2,3)-step Hamiltonian tour for $C_8 \times P_5$

Corollary 6.6 *The torus graph* $C_n \times C_m$ *is* $(2,3)$*-step Hamiltonian for (i)* $n = 3$*,* $m \equiv 0 \pmod 4$*, (ii)* $n \geq 4$*, and even* $m \geq 2$*, and (iii)* $n \equiv 0 \pmod 4$*, odd* $m \geq 3$*.*

7 Möbius Ladders

The Möbius ladder (also known as the Möbius wheel) is the cycle C_{2n}, with n additional edges joining diagonally opposite vertices as shown in Figure 24. We will denote this graph by M_{2n}, and its vertices by $v_1, v_2, \ldots, v_{2n}$, the edges of C_{2n} by v_iv_{i+1} for $i = 1, 2, \ldots, 2n \pmod{2n}$ of the cycle, and the n diagonals by v_iv_{n+i} for $i = 1, 2, \ldots, n$.

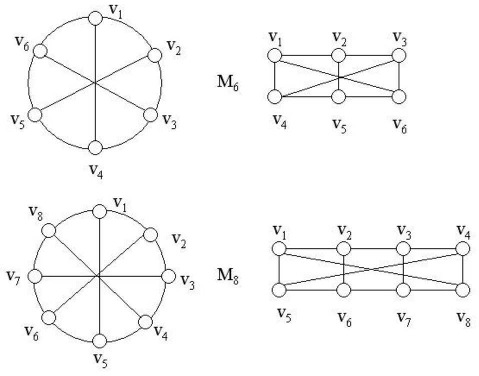
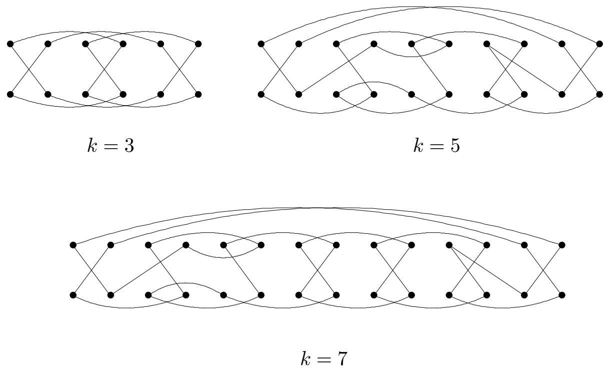


Figure 24: Möbius ladder M_6 and M_8

Theorem 7.1 *All graphs M_{2n} are $(2,3)$-step Hamiltonian if and only if $n = 2k, k \geq 3$.*

Proof. We first show that all Möbius ladders M_{4k+2} are not $(2,3)$-Hamiltonian. Observe that $D_2(M_{4k+2})$ is the disjoint union of two complete graphs of order $2k+1$. Hence, $D_2(M_{4k+2})$ does not contain a perfect matching subgraph of size $2k+1$. The theorem holds by Corollary1.3.

We now show that all Möbius ladders M_{4k} are $(2,3)$-Hamiltonian if and only if $k \geq 3$. For $k = 1, 2$, diam$(M_{4k}) < 3$ so M_{4k} is not (2,3)-Hamiltonian. For odd $k \geq 3$, we give a (2,3)-Hamiltonian tour of M_{12}, M_{20} and M_{28} (in ladder form without edges shown) in Figure 25 that can be extended to all odd $k \geq 7$. For even $k \geq 4$, we give a (2,3)-Hamiltonian tour of M_{16} and M_{24} in Figure 26 (in ladder form without edges shown) that can be extended to all even $k \geq 8$.

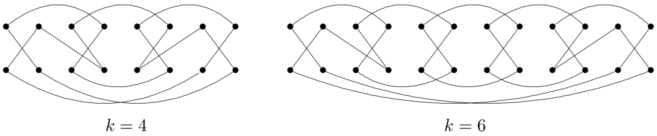


$k = 3$

$k = 5$

$k = 7$

Figure 25: A (2,3)-Hamiltonian tour for $M_{4k}, k = 3, 5, 7$.



$k = 4$

$k = 6$

Figure 26: A (2,3)-Hamiltonian tour for $M_{4k}, k = 4, 6$.

□

Theorem 7.2 *For odd $b \geq 3$ and $n \geq 2b$, if $gcd(n, (b-1)/2) = 1$ or $gcd(n, (b+1)/2) = 1$, then M_{2n} is $(1, b)$-step Hamiltonian.*

Proof. We may construct two types of Hamiltonian cycles, both shown in Figure 18 for $n = 11$ and $b = 5$. Type I may be used when $gcd(n, (b+1)/2) = 1$, for example $gcd(11, (5+1)/2) = 1$ shown in Figure 27. Type II may be used when $gcd(n, (b-1)/2) = 1$, for example $gcd(11, (5-1)/2) = 1$, also shown in Figure 18. We construct the $(1, b)$-step Hamiltonian cycle within the cycle $v_1, v_2, v_3, \ldots, v_{2n}$, where each i-th vertex is adjacent to the $(i-1)$-st, $(i+1)$-st, and $(i+n)$-th vertices, the sums considered $\pmod{2n}$, as in the Möbius ladders shown in Figure 24. Because $n \geq 2b$, the distance between the i-th and $(i+b)$-th vertices is b. For simplicity, in the following we list only the subscripts of the vertices.

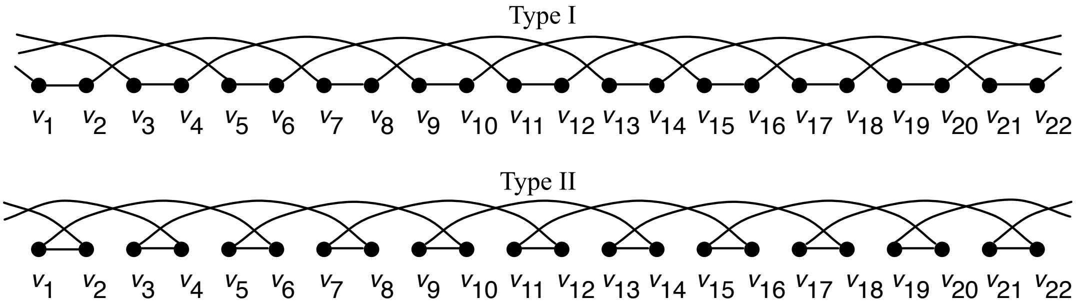


Figure 27: $(1,5)$-step Hamiltonian tours for Möbius ladder M_{22}.

Type I. The sequence of vertex subscripts in this $(1,b)$-step Hamiltonian cycle is

$$1, 2, 1+(b+1), 2+(b+1), 1+2(b+1), 2+2(b+1), \ldots, 1+(n-1)(b+1), 2+(n-1)(b+1), 1$$

where the subscripts are considered mod $2n$. Here, for $k = 0, 1, 2, \ldots, n-1$, the subscripts of the form $1 + k(b+1)$ take on all the odd values in $\{1, 2, 3, \ldots, n\}$, and the subscripts of the form $2 + k(b+1)$ take on all the even values, since $b-1$ and $b+1$ are even, as is $2n$. These subscripts are distinct, since if

$$1 + k_1(b+1) \equiv 1 + k_2(b+1) \pmod{2n},$$

then

$$(k_1 - k_2)(b+1) \equiv 0 \pmod{2n}$$

or

$$(k_1 - k_2)(b+1)/2 \equiv 0 \pmod{n}.$$

However, because $gcd(n, (b+1)/2) = 1$, we have

$$(k_1 - k_2) \equiv 0 \pmod{n}$$

which is possible only if $k_1 = k_2$. Similarly, the subscripts of the form $2 + k(b+1)$ are also distinct.

Type II. The sequence of vertex subscripts in this $(1,b)$-step Hamiltonian cycle is

$$2, 1, 2+(b-1), 1+(b-1), 2+2(b-1), 1+2(b-1), \ldots, 2+(n-1)(b-1), 1+(n-1)(b-1), 2.$$

As for Type I, similar calculations show that the vertex subscripts in this sequence are distinct.

□

Remark. If a is even, then M_{4k+2} is not (a, b)-step Hamiltonian by Corollary 1.3, since then $D_a(M_{4k+2})$ has two or more components with $2k+1$ or fewer vertices. For example, $D_4(M_{18})$ is composed of two disjoint 9-vertex components, each containing one C_9 and three disjoint copies of C_3.

Problem 7.1 *For what values of b and odd a is M_{4k+2} (a, b)-step Hamiltonian?*

Theorem 7.3 *For even $b \geq 2$ and $n \geq 2b$, if $gcd(n, b/2) = 1$, then M_{2n} is $(1, b)$-step Hamiltonian.*

Proof. Figure 28 shows an example for $n = 14, b = 6$, where $gcd(14, 6/2) = 1$. The sequence of vertex subscripts in general in the $(1, b)$-step Hamiltonian tour is:

$$2, 1, 1+b, 2+b, 2+2b, 1+2b, 1+3b, 2+3b, \ldots, 1+(n-1)b, 2+(n-1)b, 2$$

where the subscripts are considered mod $2n$. The subscripts may be shown to be distinct as in the proof of Theorem 7.2.

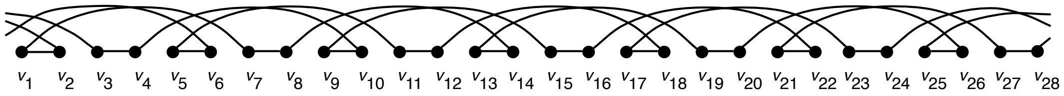


Figure 28: $(1, 6)$-step Hamiltonian tours for Möbius ladder M_{28}.

□

Corollary 7.4 *All graphs M_{2n} are (i) $(1, 2)$-step Hamiltonian if $n = 2k, k \geq 2$; (ii) $(1, 3)$-step Hamiltonian if $n \geq 5$.*

References

[1] G.C. Lau, S.M. Lee, K. Schaffer, S.M. Tong and S. Lui, On k-step Hamiltonian graphs, *JCMCC*, in press.

[2] G.C. Lau, S.M. Lee, K. Schaffer and S.M. Tong, On k-step Hamiltonian bipartite and tripartite graphs, *Malaya J. Mat.*, in press.

[3] G.C. Lau, S.M. Lee, K. Schaffer and S.M. Tong, A complete solution on 3-step Hamiltonian grids, *submitted*, 2014.

[4] W. T. Tutte, On Hamiltonian circuits, Journal of the London Mathematical Society (2nd ser.) 21 (2) (1946), 98-101.

[5] Weisstein, Eric W. Star Polygon. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/StarPolygon.html, accessed June 23, 2017.